Bayesian Kernel Shaping for Learning Control: Appendix

The Complete Log Likelihood

The complete log likelihood L is:

$$
\begin{aligned}
L &= \log \left[\prod_{i=1}^{N} \left[\left[p(y_i|\mathbf{z}_i, \sigma^2) p(\mathbf{z}_i|\mathbf{x}_i, \mathbf{b}, \psi_z) \right]^{w_i} \prod_{m=1}^{d} p(w_{im}) \right] \prod_{m=1}^{d} p(\mathbf{b}_m|\psi_{zm}) p(\psi_{zm}) p(h_m) p(\sigma^2) \right] \\
&= \sum_{i=1}^{N} \log p(y_i|\mathbf{z}_i, \sigma^2)^{w_i} + \sum_{i=1}^{N} \log p(\mathbf{z}_i|\mathbf{x}_i, \mathbf{b}, \psi_z)^{w_i} + \sum_{i=1}^{N} \sum_{m=1}^{d} \log p(w_{im}) + \sum_{m=1}^{d} \log p(h_m) \\
&\quad + \sum_{m=1}^{d} \log p(\mathbf{b}_m|\psi_{zm}) + \sum_{m=1}^{d} \log p(\psi_{zm}) + \log p(\sigma^2)
\end{aligned} \tag{1}
$$

We can apply a variational approach on concave/convex functions in order to lower bound a problematic $\log \left(1 + (x_{im} - x_{qm})^{2r} h_m \right)$ term that prevents us from deriving an analytically tractable posterior distribution for h_m:

$$
\begin{aligned}
&\sum_{i=1}^{N} \sum_{m=1}^{d} \log p(w_{im}) \\
&= \sum_{i=1}^{N} \sum_{m=1}^{d} w_{im} \log \left(\frac{1}{1 + (x_{im} - x_{qm})^{2r} h_m} \right) \\
&\quad + \sum_{i=1}^{N} \sum_{m=1}^{d} (1 - w_{im}) \log \left(1 - \frac{1}{1 + (x_{im} - x_{qm})^{2r} h_m} \right) \\
&= \sum_{i=1}^{N} \sum_{m=1}^{d} w_{im} \log \left(\frac{1}{1 + (x_{im} - x_{qm})^{2r} h_m} \right) + \sum_{i=1}^{N} \sum_{m=1}^{d} (1 - w_{im}) \log \frac{(x_{im} - x_{qm})^{2r} h_m}{1 + (x_{im} - x_{qm})^{2r} h_m} \\
&= \sum_{i=1}^{N} \sum_{m=1}^{d} (1 - w_{im}) \log (x_{im} - x_{qm})^{2r} h_m - \sum_{i=1}^{N} \sum_{m=1}^{d} \log \left(1 + (x_{im} - x_{qm})^{2r} h_m \right) \\
&\geq \sum_{i=1}^{N} \sum_{m=1}^{d} (1 - w_{im}) \log (x_{im} - x_{qm})^{2r} h_m - \sum_{i=1}^{N} \sum_{m=1}^{d} \lambda_{im} (x_{im} - x_{qm})^{2r} h_m
\end{aligned} \tag{2}
$$

where λ_{im} is a variational parameter to be optimized.

Incorporating the lower bound approximation in Eq. 2 into the expression for the complete log likelihood L in Eq. 1, the resulting lower bound to L is then $\hat{L}$:

$$
\begin{aligned}
\hat{L} \geq & -\sum_{i=1}^{N} w_i \log 2\pi\sigma^2 - \sum_{i=1}^{N} \frac{w_i \left(y_i - \mathbf{1}^T \mathbf{z}_i\right)^2}{2\sigma^2} - \sum_{i=1}^{N}\sum_{m=1}^{d} w_i \log 2\pi\psi_{zm} \\
& - \sum_{i=1}^{N}\sum_{m=1}^{d} \frac{w_i \left(z_{im} - \mathbf{b}_m^T \mathbf{x}_{im}\right)^2}{2\psi_{zm}} + \sum_{i=1}^{N}\sum_{m=1}^{d} \left(1 - w_{im}\right) \log \left(x_{im} - x_{qm}\right)^{2r} h_m \\
& - \sum_{i=1}^{N}\sum_{m=1}^{d} \lambda_{im} \left(x_{im} - x_{qm}\right)^{2r} h_m - \frac{1}{2}\sum_{m=1}^{d} \log \left|\mathbf{\Sigma}_{\mathbf{b}_m,0}\right| - \frac{1}{2}\sum_{m=1}^{d} \log \psi_{zm} \\
& - \sum_{m=1}^{d} \frac{\mathbf{b}_m^T \mathbf{\Sigma}_{\mathbf{b}_m,0}^{-1} \mathbf{b}_m}{2\psi_{zm}} - \sum_{m=1}^{d} \left(\frac{n_{m0}}{2} + 1\right) \log \psi_{zm} - \sum_{m=1}^{d} \frac{n_{m0}\psi_{zmN0}}{2\psi_{zm}} \\
& + \sum_{m=1}^{d} \left(a_{hm0} - 1\right) \log h_m - \sum_{m=1}^{d} b_{hm0} h_m + \text{const}
\end{aligned} \tag{3}
$$

Final Posterior EM Update Equations

E-step:

$$
\mathbf{\Sigma}_{\mathbf{b}_m} = \left(\mathbf{\Sigma}_{\mathbf{b}_m,0}^{-1} + \sum_{i=1}^{N} \langle w_i \rangle \mathbf{x}_{im} \mathbf{x}_{im}^T\right)^{-1} \tag{4}
$$

$$
\langle \mathbf{b}_m \rangle = \mathbf{\Sigma}_{\mathbf{b}_m} \left(\sum_{i=1}^{N} \langle w_i \rangle \langle \mathbf{z}_{im} \rangle \mathbf{x}_{im}\right) \tag{5}
$$

$$
\mathbf{\Sigma}_{\mathbf{z}_i|y_i,\mathbf{x}_i} = \frac{\mathbf{\Psi}_{zN}}{\langle w_i \rangle} - \frac{1}{s_i}\left(\frac{\mathbf{\Psi}_{zN}}{\langle w_i \rangle}\mathbf{1}\mathbf{1}^T \frac{\mathbf{\Psi}_{zN}}{\langle w_i \rangle}\right) \tag{6}
$$

$$
\langle \mathbf{z}_i \rangle = \frac{\mathbf{\Psi}_{zN}}{s_i \langle w_i \rangle}\mathbf{1} + \left(\mathbf{I}_{d,d} - \frac{\mathbf{\Psi}_{zN}}{s_i \langle w_i \rangle}\mathbf{1}\mathbf{1}^T\right) \mathbf{b}\mathbf{x_i} \tag{7}
$$

$$
\psi_{zmN} = \frac{\left(\begin{array}{c} \sum_{i=1}^{N} \langle w_i \rangle \left(\langle z_{im} \rangle - \langle \mathbf{b}_m \rangle^T \mathbf{x}_{im}\right)^2 + \sum_{i=1}^{N} \langle w_i \rangle \mathbf{\Sigma}_{\mathbf{z}_i|y_i,\mathbf{x}_i} + n_{m0}\psi_{zmN0} \\ + \langle \mathbf{b}_m \rangle^T \mathbf{\Sigma}_{\mathbf{b}_m,0}^{-1} \langle \mathbf{b}_m \rangle \end{array}\right)}{n_{m0} + \sum_{i=1}^{N} \langle w_i \rangle} \tag{8}
$$

$$
\langle w_{im} \rangle = \frac{q_{im} A_i^{\prod_{k=1,k\neq m}^{d} \langle w_{ik} \rangle}}{q_{im} A_i^{\prod_{k=1,k\neq m}^{d} \langle w_{ik} \rangle} + 1 - q_{im}} \tag{9}
$$

$$
\langle h_m \rangle = \frac{a_{hm0} + N - \sum_{i=1}^{N} \langle w_{im} \rangle}{b_{hm0} + \sum_{i=1}^{N} \lambda_{im} \left(x_{im} - x_{qm}\right)^{2r}} \tag{10}
$$

M-step:

$$
\sigma^2 = \frac{1}{\sum_{i=1}^{N} \langle w_i \rangle} \sum_{i=1}^{N} \left\langle w_i \left(y_i - \mathbf{1}^T \mathbf{z}_i\right)^2 \right\rangle \tag{11}
$$

$$
\lambda_{im} = \frac{1}{1 + \left(x_{im} - x_{qm}\right)^{2r} h_m} \tag{12}
$$

where:

$$\begin{aligned}
\langle w_i \rangle &= \prod_{m=1}^{d} \langle w_{im} \rangle \\
s_i &= \sigma^2 + \mathbf{1}^T \frac{\mathbf{\Psi}_{zN}}{\langle w_i \rangle} \mathbf{1} \\
q_{im} &= \frac{1}{1 + (x_{im} - x_{qm})^{2r} \langle h_m \rangle} \\
A_i &= \text{Normal}\left(y_i; \mathbf{1}^T \langle \mathbf{z}_i \rangle, \sigma^2\right) \prod_{m=1}^{d} \text{Normal}\left(z_{im}; \langle \mathbf{b}_m \rangle^T \mathbf{x}_{im}, \psi_{zm}\right)
\end{aligned}$$

and $\mathbf{I}_{d,d}$ is a $d \times d$ identity matrix, $\mathbf{bx_i}$ is a d by 1 vector with coefficients $\langle \mathbf{b}_m \rangle^T \mathbf{x}_{im}$, $\mathbf{\Psi}_{zN}$ is a diagonal matrix with ψ_{zN} on its diagonal,